Modelling and Control of Flexible Link Multi-Body Systems:

Port-Hamiltonian Based Modelling.

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Overview

Introduction

The port-Hamiltonian formalism brings together:

- Port-based modeling approach (bond-graph);
- Geometric mechanics (coordinate-free);
- Systems and control theory.

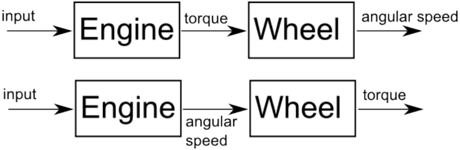
Bond graphs provide a unified framework for the modeling of physical systems

Henry Paynter (1959)

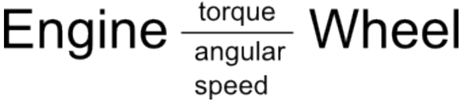
- Different domains (mechanical, electrical, hydraulic, thermal)
- Energy is the *lingua franca*;
- Complex systems are written as a composition of ideal components: energy-storage, energy-dissipation, energy-routing, etc.

Bond graph vs block diagram

Block diagram

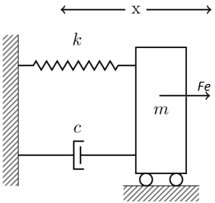


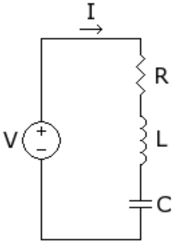
Bond Graph

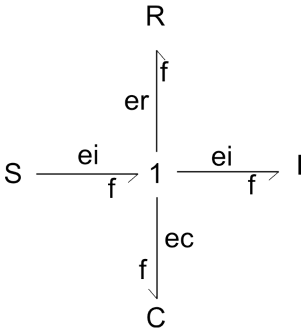


“Bonds” are energy-conserving and acausal!

Bond graphs provides a unified language for systems of different physical domains

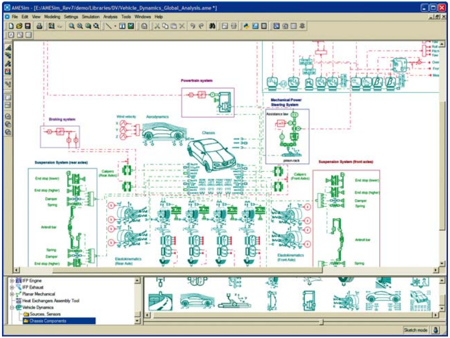






	Mass-spring-damper	RLC
Flow variables (f)	speed	current
Effort variables (e)	force	voltage
I storage (I)	mass (Inertia)	inductor
C storage (C)	spring	capacitor
Dissipation element (R)	damper	resistor

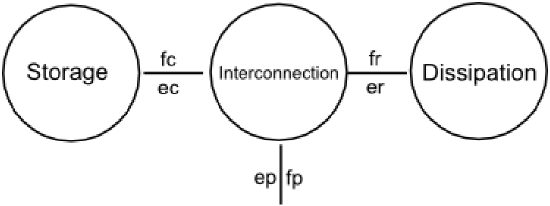
Bond graphs are used to model COMPLEX systems



Many commercial software based on bond-graphs:

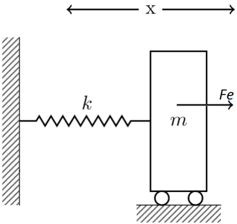
- AMEsim
- 20-sim (U. Twente)
- Simscape (Matlab/Simulink)
- Dymola (Catia / Dassault)

Port-Hamiltonian systems = bond graph elements + Hamiltonian



- Storage function is the *Hamiltonian* (energy) $x \mapsto H(x)$
- Storage *flow* variable: $f_c := -\dot{x}$,
- Storage *effort* variable: $e_c := \frac{\partial H}{\partial x}(x) = \boldsymbol{grad}_x H(x)$,
- Energy flow: $\dot{H}(t) = \frac{\partial^T H}{\partial x}\dot{x} = -e_c^T f_c = e_p^T f_p + e_r^T f_r$

Simple example of port-Hamiltonian system



From Newton 2nd Law:

$$m\ddot{x} + kx = F_{ext},$$

System energy:

$$E = \frac{m}{2}\dot{x}^2 + \frac{k}{2}x^2$$

By choosing $p = m\dot{x}$ **then** $H(p,x) = \frac{1}{2m}p^2 + \frac{k}{2}x^2$

$$\begin{bmatrix} \dot{p} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{p}{m} \\ kx \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} F_{ext},$$

$$\dot{x} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{p}{m} \\ kx \end{bmatrix}$$

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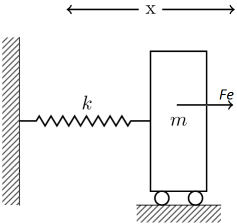

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$$\underbrace{\begin{bmatrix}\dot{p}\\ \dot{x}\end{bmatrix}}_{-f_c} = \underbrace{\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}}_{J}\underbrace{\begin{bmatrix}\partial_p H\\ \partial_x H\end{bmatrix}}_{e_c} + \begin{bmatrix}1\\ 0\end{bmatrix}\underbrace{F_{ext}}_{e_p},$$

$$\underbrace{\dot{x}}_{f_p} = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}\partial_p H\\ \partial_x H\end{bmatrix}$$

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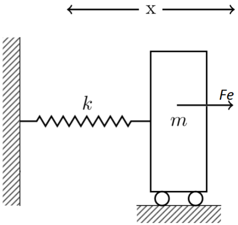

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$$\underbrace{\dot{x}}_{f_p} = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}\partial_p H\\ \partial_x H\end{bmatrix}$$

$\Longrightarrow \dot{H} = F_{ext}\dot{x}$

F_{ext} and $\dot{x}$ are the interconnection ports.

Typical mathematical representation of finite-dimensional port-Hamiltonian systems

$$\dot{x} = J\frac{\partial H}{\partial x} + Bu,$$
$$y = B^T\frac{\partial H}{\partial x}.$$

where:
$H(x)$: system Hamiltonian;
$x \in \mathbb{R}^n$: energy variables;
$u \in \mathbb{R}^m$: inputs;
$y \in \mathbb{R}^m$: outputs;
J: interconnection matrix (skew-symmetric);

$$\Longrightarrow \dot{H} = y^T u$$

The interconnection of two (or N) pHs is still a pHs

Individual systems

$$\dot{x}_1 = J_1 \frac{\partial H_1}{\partial x_1} + B_1 u_1,$$
$$y_1 = B_1^T \frac{\partial H_1}{\partial x_1},$$

$$\dot{x}_2 = J_2 \frac{\partial H_2}{\partial x_2} + B_2 u_2,$$
$$y_2 = B_2^T \frac{\partial H_2}{\partial x_2},$$

Coupled system

Interconnectic $H(x_1, x_2) = H_1(x_1) + H_2(x_2)$

$$u_1 = y_2 + u_e,$$
$$u_2 = -y_1$$

$$\Longrightarrow$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} J_1 & B_1 B_2^T \\ -B_2 B_1^T & J_2 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial x_1} \\ \frac{\partial H}{\partial x_2} \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u_e,$$
$$y_e = \begin{bmatrix} B_1^T & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial x_1} \\ \frac{\partial H}{\partial x_2} \end{bmatrix}$$

$$\frac{dH}{dt} = y_e^T u_e$$

PHSs are convenient for (non-linear) control design

Typical port-Hamiltonian representation:

$$\dot{x} = J\frac{\partial H}{\partial x}(x) + Bu,$$
$$y = B^T\frac{\partial H}{\partial x}(x).$$

We've seen that: $\dot{H} = y^T u$

What happens if $u = -K(x)y$, with $K(x) > 0$?

$$\dot{H} = -y^T K(x) y \leq 0,$$

If $H(x)$ is lower bounded, the controlled system is stable!

General ideas on Port Hamiltonian Systems (pHs)

1. strongly structured mathematical dynamical systems: both linear and non-linear, both finite-dimensional and infinite-dimensional,
2. based on physical grounds, allowing for different modelling levels,
3. all physics permitted: solid mechanics, structural mechanics, fluid mechanics, electromagnetism, electrical circuits, ...
4. comes along with specific numerical methods, which do preserve, at the discrete level, the structure of the continuous equations,
5. allows for open dynamical systems, with interacting ports,
6. modularity: interconnection of sub-systems, e.g. Multi-Body Systems, and... easy multiphysics modelling, e.g. Fluid-Structure Interaction,
7. physically-based strategy for control and stabilization[1],
8. extensions to dissipative dynamical systems are available.

[1] wait till... Part II, by Paul Kotyczka, this afternoon!

Outline

Overview

Useful Tools in Finite Dimension

These 3 tools will be of major help in the study:

- the **gradient**: to be able to compute $\boldsymbol{grad}_X H(X)$, when $X \in \mathbb{R}^{2n}$,
- **skew symmetric** matrices: $J^T = -J$,
- in the special case of quadratic Hamiltonian, hence linear dynamical systems $\dot{X} = A\,X$: **matrix exponentials.**

  Straightforward consequences on **stability**:

  1. $\mathrm{spec}(A) \in \mathbb{C}_0^- \implies (\exp(t\,A) \to 0$, as $t \to \infty)$, i.e. asymptotic stability.
  2. $\exists \lambda \in \mathrm{spec}(A),\ \Re e(\lambda) > 0 \implies (\|\exp(t\,A)\| \to \infty$, as $t \to \infty)$, i.e. (exponential) instability.
  3. the case with eigenvalues on $i\mathbb{R}$ is more subtle, and requires geometric insight on the eigenspaces to be solved: either stability or (polynomial) instability can be found.

Ex 0: a 1-d.o.f.linear oscillator (1)

Original dynamics: $m\ddot{x} + kx = 0\,,$ with $(x_0, \dot{x}_0)$ initial data , (1)

usually rewritten in state-space form, as:

$$\frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{m} \\ -k & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix}\,, \quad \text{with } \begin{bmatrix} x_0 \\ \dot{x}_0 \end{bmatrix} \text{ initial data}\,. \tag{2}$$

$\Longrightarrow$ Choose the **Hamiltonian formalism**!

- **Energy variables**: position $q := x$, momentum $p := m\,\dot{x}$,

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- **Energy variables**: position $q := x$, momentum $p := m\,\dot{x}$,
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- **Co-energy variables**: $\partial_q H = k\,q$ force, $\partial_p H = \frac{1}{m}\,p := \dot{x}$ velocity,

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- **Hamiltonian function**: $H(X) := \frac{1}{2\,m}\,p^2 + \frac{1}{2}k\,q^2$, with $X := (q,\,p)$,
- **Co-energy variables**: $\partial_q H = k\,q$ force, $\partial_p H = \frac{1}{m}\,p := \dot{x}$ velocity,
- **Dynamical system**:

$$\frac{d}{dt}X = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} k\,q \\ \frac{1}{m}\,p \end{bmatrix} = J\,\boldsymbol{grad}_X H(X)\,.$$

with skew-symmetric matrix $J := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, i.e. $J^T = -J$.

Ex 0: a 1-d.o.f.linear oscillator (2)

Theorem

J skew-symmetric $\Longrightarrow \frac{d}{dt}H(X(t)) = 0$.
Hence, the dynamical system is conservative, i.e. $H(X(t)) = H(X_0)$.

$$\begin{aligned} \frac{dH}{dt}(t) &= (\boldsymbol{grad}_X H(X),\, \frac{d}{dt}X)_{\mathbb{R}^2} \\ &= (\boldsymbol{grad}_X H(X),\, J\,\boldsymbol{grad}_X H(X))_{\mathbb{R}^2} \\ &= 0\,, \quad \text{since } J \text{ is skew symmetric!} \end{aligned}$$

■

Ex 1: the n-d.o.f. linear oscillator

Original dynamics: $M\ddot{x} + Kx = 0\,,$ with $(x_0, \dot{x}_0)$ initial data , (3)

with mechanical parameters $M = M^T > 0$, $K = K^T \geq 0$.

- **Energy variables**: $q := x \in \mathbb{R}^n$, $p := M\,\dot{x} \in \mathbb{R}^n$, set $X = (q,\, p) \in \mathbb{R}^{2n}$,
- **Hamiltonian function**: $H(X) := \frac{1}{2} p^T\, M^{-1}\, p + \frac{1}{2} q^T\, K\, q$,
- **Co-energy variables**: $\boldsymbol{grad}_q H = K\,q$, and $\boldsymbol{grad}_p H = M^{-1}\,p := \dot{x}$,
- **Dynamical system** in standard form:

$$\frac{d}{dt} X = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} K\,q \\ M^{-1}\,p \end{bmatrix} = J\,\boldsymbol{grad}_X H(X)\,,$$

with $J := \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$ is skew-symmetric in $\mathbb{R}^{2n}$.

Ex 2: a 1-d.o.f. nonlinear oscillator: the pendulum

Original dynamics: $J\,\ddot{\theta} + g\,\sin(\theta) = 0$, with $(\theta_0, \dot{\theta}_0)$ initial data. (4)

$\Longrightarrow$ Choose the **Hamiltonian formalism**!

- **Energy variables**: position $q := \theta$, momentum $p := J\,\dot{\theta}$,
- **Hamiltonian function**: $H(X) := \frac{1}{2J}\,p^2 + g\,(1 - \cos(\theta))$,
- **Co-energy variables**: $\partial_q H = g\,\sin(\theta)$ torque, $\partial_p H = \frac{1}{J}\,p := \dot{\theta}$ angular velocity,
- **Dynamical system** with $X = (q,\,p)$:

$$\frac{d}{dt}X = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} g\,\sin(\theta) \\ \dot{\theta} \end{bmatrix} = J\,\boldsymbol{grad}_X H(X)\,.$$

Theorem

J skew-symmetric $\Longrightarrow \frac{d}{dt}H(X(t)) = 0$. Hence, the **non-linear** *system is conservative, i.e. $H(X(t)) = H(X_0)$, with* **non-quadratic** *Hamiltonian.*

Open Systems, Ports, and Energy Balance (1)

Suppose we have **interaction** with the environment, by means of:

- actuators, with control $u \in \mathbb{R}^m$,
- sensors, with co-localized measurements or observations $y \in \mathbb{R}^m$,

then, the port-Hamiltonian system (pHs) is defined by:

$$\dot{X} = J\,\mathbf{grad}_X H(X) + g(X)\,u(t)\,, \quad (5)$$

$$y(t) = g(X)^T\,\mathbf{grad}_X H(X)\,. \quad (6)$$

Theorem

J skew-symmetric $\Longrightarrow$ the system is lossless. Indeed,
$\frac{d}{dt}H(X(t)) = (y(t), u(t))_{\mathbb{R}^m}$, or $H(X(t)) = H(X_0) + \int_0^t (y(\tau), u(\tau))_{\mathbb{R}^m}\,\mathrm{d}\tau$.

What about the linear / quadratic case?

Suppose the Hamiltonian function is quadratic $H(X) := \frac{1}{2}X^T\,Q\,X$, with $Q = Q^T > 0$, we then easily compute $\boldsymbol{grad}_X H(X) = QX$, and we can define the closed linear dynamical system:

$$\dot{X} = J\,Q\,X\,,$$

that is, the **matrix of dynamics** reads $A := J\,Q$.

Let $B := g$ be the **control matrix** of size $n \times m$, then the open dynamical system is given by:

$$\begin{aligned} \dot{X} &= J\,Q\,X + B\,u(t)\,, \qquad (7) \\ y(t) &= B^T\,Q\,X\,. \qquad (8) \end{aligned}$$

that is, the $m \times n$ **observation matrix** reads $C := B^T\,Q$.

Ex 1: n-d.o.f. damped oscillator

Original dynamics: $M\ddot{x} + (C + G)\dot{x} + Kx = 0$, with $(x_0, \dot{x}_0)$ initial data, (9)

with $M = M^T > 0$, $K = K^T \geq 0$ and $C = C^T$, $G = -G^T$.

- Energy variables: $q := x \in \mathbb{R}^n$, $p := M\,\dot{x} \in \mathbb{R}^n$,
- Hamiltonian function: $H(X) = \frac{1}{2} p^T M^{-1} p + \frac{1}{2} q^T K\, q$,
- Dynamical system in standard form:

$$\frac{d}{dt}X = \begin{bmatrix} 0 & I \\ -I & -(G + C) \end{bmatrix} \boldsymbol{grad}_X H(X) = (J - R)\, \boldsymbol{grad}_X H(X)\,,$$

with $J := \begin{bmatrix} 0 & I \\ -I & -G \end{bmatrix}$, and $R := \begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix}$.

$\Longrightarrow$ Examine the two terms separately :

- Role of the skew-symmetric G matrix, the gyroscopic term.
- Damping effect is ensured, provided that $C = C^T \geq 0$.

Ex 1: about the G matrix

1. This matrix is often not considered in modelling processes of damping! Why?
   When $C = 0$, whatever $G = -G^T$, the system proves conservative: $\frac{d}{dt} H_0(X(t)) = 0$!
2. Is it a naive generalizations due to mathematicians? No!
   Classical mechanical example: Coriolis force $f = \omega \wedge \dot{x}$, with rotational speed $\omega = (p, q, r)^T$, $f = G_\omega \dot{x}$ where
   $G_\omega := \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix}$ is skew symmetric.
3. Also in electromagnetics: Lorentz force in a uniform magnetic field.

Open Systems with Damping, Energy Balance (2)

The port-Hamiltonian system (pHs) is defined by:

$$\dot{X} = (J - R)\,\boldsymbol{grad}_X H(X) + g(X)\,u(t)\,, \quad (10)$$

$$y(t) = g(X)^T\,\boldsymbol{grad}_X H(X)\,. \quad (11)$$

Theorem

J skew-symmetric and R positive $\Longrightarrow$ the system is passive. Indeed,

$$\frac{d}{dt}H(X(t)) = -(\boldsymbol{grad}_X H(X), R\,\boldsymbol{grad}_X H(X))_{\mathbb{R}^n} + (y(t), u(t))_{\mathbb{R}^m}\,,$$

$$\leq (y(t), u(t))_{\mathbb{R}^m}\,.$$

Hence, $H(X(t)) \leq H(X_0) + \int_0^t (y(\tau), u(\tau))_{\mathbb{R}^m}\,\mathrm{d}\tau$.

Overview

1. Introduction
2. Finite-dimensional case: Ordinary Differential Equations (ODEs)
   - Useful Tools
   - Closed Systems
   - Open Systems
   - A short word on dissipation?
3. **Infinite-dimensional case: Partial Differential Equations (PDEs)**
   - New Useful Tools
   - Closed systems
   - Open systems
4. Outlook

Useful Notions in Infinite Dimension

These notions will be of major help in the following:

- **functions** u, instead of vectors X,
- an infinite-dimensional **Hilbert functional space** $\mathcal{H}$ for functions, instead of a finite-dimensional Euclian vector space $\mathbb{R}^{2n}$,
- a Hamiltonian **functional** H, defined on functions u, instead of a Hamiltonian function defined on vectors, e.g.:

$$\begin{aligned} H : \mathcal{H} &\rightarrow \mathbb{R} \\ u &\mapsto H(u) := \frac{1}{2} \int_0^L u(z)^2 \, \mathrm{d}z \end{aligned}$$

Useful Tools in Infinite Dimension

These tools will be of major help in the following:

- the **variational derivative** of a functional: $\delta_u H$, in place of the gradient of the function, defined by

$$H(u + \varepsilon\, w) = H(u) + \varepsilon\, (\delta_u H,\, w)_{\mathcal{H}} + O(\varepsilon^2)$$

  N.B. in the above easy example, $\delta_u H = u$.

- formally **skew symmetric operators**: $\mathcal{J}^T = -\mathcal{J}$, w.r.t the scalar product in the Hilbert space $\mathcal{H}$, i.e.

$$(u,\, \mathcal{J} v)_{\mathcal{H}} = -(\mathcal{J} u,\, v)_{\mathcal{H}}$$

- in the special case of quadratic Hamiltonian, hence linear dynamical systems $\dot{X} = \mathcal{A} X$: **semigroups.**

Definitions for (linear) pHs (1)

Consider the **dynamical system**

$$\frac{d}{dt}X(z,t) = \mathcal{J}\,\delta_X \mathcal{H}(X) \tag{12}$$

with (quadratic) **Hamiltonian** functional:

$$\mathcal{H}(X) = \frac{1}{2}\int_0^L X(z,t)^T \mathcal{L} X(z,t)\,\mathrm{d}z\,,$$

its (linear) **variational derivative** reads:

$$\delta_X \mathcal{H}(X) = \mathcal{L} X(z,t)\,.$$

We suppose that operator $\mathcal{J}$ is formally skew-symmetric.

Indeed in the linear case, we get back to **semigroups**, with $\mathcal{A} = \mathcal{J}\mathcal{L}$:

$$\frac{d}{dt}X(z,t) = (\mathcal{J}\mathcal{L})\,X(z,t)\,.$$

Definitions for pHs (2)

The energy balance associated to this system is:

$$\begin{aligned} \frac{d\mathcal{H}}{dt}(t) &= \int_0^L \delta_X \mathcal{H}(X).\frac{dX}{dt}\,\mathrm{d}z\,, \\ &= \int_0^L \delta_X \mathcal{H}(X).\mathcal{J}.\delta_X \mathcal{H}(X)\,\mathrm{d}z\,, \\ &= 0\,, \quad \text{since } \mathcal{J} \text{ is skew-adjoint?} \quad \text{Almost!} \end{aligned}$$

Be careful, $\mathcal{J}$ is skew-adjoint, but only... formally.
$\Longrightarrow$ Energy flows through the boundary, only!

There is no internal dissipation of any kind in this case.

$\Longrightarrow$ Let us compute things more concretely on two examples:

1. Webster horn equation (linear, space varying coefficients)
2. Euler-Bernoulli beam equation (linear, constant coefficients)

Ex 3: Webster horn equations (1)

Consider an axi-symmetric horn with varying cross-section $z \mapsto S(z)$.

- **energy variables**: density ρ, and particle velocity v,
- with pressure $p := c_0^2\,\rho$, and energy density $U(\rho) := \frac{c_0^2}{2\,\rho_0^2}\rho^2$,

Hamiltonian $$H(\rho,v) := \int_0^L \left(\frac{1}{2}\rho_0 v^2 + \rho_0 U(\rho)\right) S(z)\,\mathrm{d}z,$$

- **co-energy variables**: $\delta_\rho H = \rho_0\,U'(\rho) = \frac{1}{\rho_0}\,p$ and $\delta_v H = \rho_0\,v$,
- the **dynamical system** reads:

$$\frac{d}{dt}\begin{bmatrix}\rho\\ v\end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{S}\partial_z(S\,\cdot\,) \\ -\partial_z & 0\end{bmatrix}\begin{bmatrix}\delta_\rho H\\ \delta_v H\end{bmatrix}. \qquad (13)$$

Proposition

Operator $\mathcal{J}$ is formally skew-symmetric, w.r.t the scalar product

$$(\boldsymbol{e},\boldsymbol{f})_{\mathcal{H}} := \int_0^L (e_1 f_1 + e_2 f_2)\,S(z)\,\mathrm{d}z\,.$$

Ex 3: Webster horn equations (2)

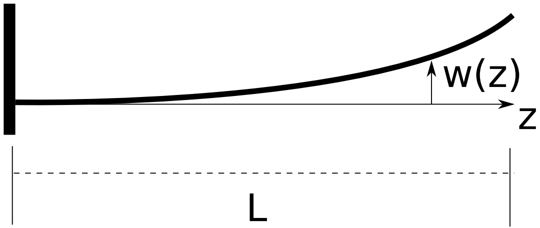
Indeed, for the above horn equation, we can compute:

$$\begin{aligned}
\frac{d\mathcal{H}}{dt}(t) &= \int_0^L (e_1 f_1 + e_2 f_2)\, S(z)\, \mathrm{d}z\,, \quad \text{with } \boldsymbol{f} = \mathcal{J}\, \boldsymbol{e}\,, \\
&= \int_0^L (-e_1\, \partial_z (S e_2) - S e_2\, \partial_z e_1)\, \mathrm{d}z\,, \\
&= \int_0^L -\partial_z (e_1\, S e_2)\, \mathrm{d}z\,, \\
&= S(0) e_1(0)\, e_2(0) - S(L) e_1(L)\, e_2(L)\,, \\
&= p(0).S(0) v(0) - p(L).S(L) v(L)\,. \\
&= p(0).u(0) - p(L).u(L)\,.
\end{aligned}$$

$\Longrightarrow$ Energy flows through the boundary, only!

Here $u := Sv$ is volume velocity, and $p.u$ is also meaningful.

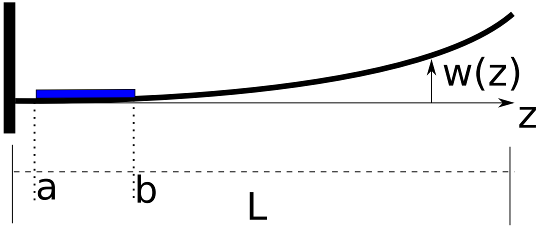
The piezoelectric bending beam equations

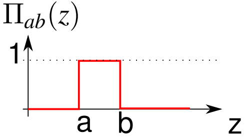


Classical Euler-Bernoulli equations:

$$\rho S \frac{\partial^2}{\partial t^2} w(z,t) = - \frac{\partial^2}{\partial z^2} \left(EI \frac{\partial^2}{\partial z^2} w \right) .$$

The piezoelectric bending beam equations

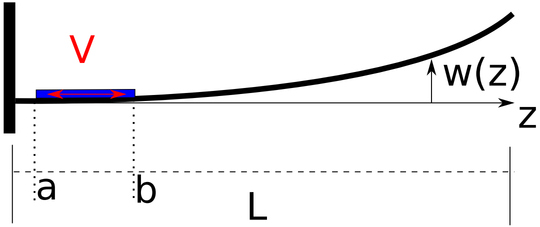


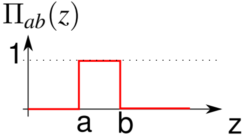


Including the piezoelectric patch rigidity and mass:

$$\underbrace{(\rho S + \Pi_{ab}(z)\rho_p S_p)}_{\mu(z)} \frac{\partial^2}{\partial t^2} w(z,t) = -\frac{\partial^2}{\partial z^2}\left(\underbrace{(EI + \Pi_{ab}(z) E_p I_p)}_{\kappa(z)} \frac{\partial^2}{\partial z^2} w\right) .$$

The piezoelectric bending beam equations

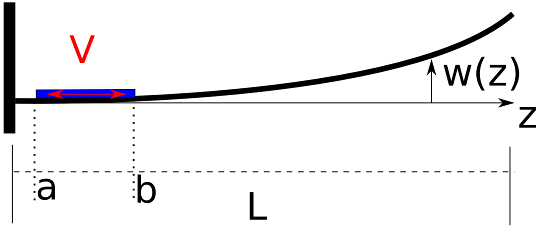


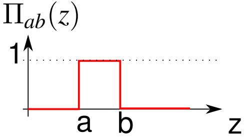


Including the moment due to voltage:

$$\underbrace{(\rho S + \Pi_{ab}(z)\rho_p S_p)}_{\mu(z)} \frac{\partial^2}{\partial t^2} w(z,t) = -\frac{\partial^2}{\partial z^2}\left(\underbrace{(EI + \Pi_{ab} E_p I_p)}_{\kappa(z)} \frac{\partial^2}{\partial z^2} w + \Pi_{ab}(z) k_p v(z,t) \right) .$$

The piezoelectric bending beam equations





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$$\mu(z) \frac{\partial^2}{\partial t^2} w(z,t) = -\frac{\partial^2}{\partial z^2}\left(\kappa(z)\frac{\partial^2}{\partial z^2} w\right) + \underbrace{\frac{\partial^2}{\partial z^2}\left(\Pi_{ab}(z)\right.}_{\text{unbounded input operator}} \left. k_p v(z,t)\right) .$$

. Modeling with piezo, see BANKS, SMITH and WANG 1996 ”Smart Material Structures” (Chapter 4), Wiley

Ex 4: Bending eqn as port-Hamiltonian system (1)

Let us define the energy variables $x_1(z,t)$ and $x_2(z,t)$ as:

$$x_1(z,t) := \mu(z)\dot{w}(z,t) = \text{linear momentum}$$
$$x_2(z,t) := \partial^2_{z^2} w(z,t) = \text{strain}\,.$$

The system Hamiltonian is given by:

$$H[x_1,x_2] = \frac{1}{2}\int_{z=0}^{L}\left(\frac{x_1(z,t)^2}{\mu(z)} + \kappa(z)x_2(z,t)^2\right)\mathrm{d}z\,,$$

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The co-energy variables are the variational derivatives of the Hamiltonian with respect to x_1 and x_2:

$$e_1(z,t) := \frac{\delta H}{\delta x_1} = \frac{x_1(z,t)}{\mu} = \dot{w}(z,t) = \text{vertical speed}$$
$$e_2(z,t) := \frac{\delta H}{\delta x_2} = \kappa(z)x_2(z,t) = \kappa(z)\partial^2_{z^2} w = \text{restoring torque}\,.$$

Ex 4: Bending eqn as port-Hamiltonian system (2)

The piezoelectric beam equation can thus be rewritten as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -\partial^2_{z^2} \\ \partial^2_{z^2} & 0 \end{bmatrix}}_{\mathcal{J}} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} + \begin{bmatrix} \partial^2_{z^2} \\ 0 \end{bmatrix} \Pi_{ab}(z) k_p v(z,t) \,,$$

where $\mathcal{J}$ is a formally skew-symmetric operator on $L^2(0,L)$.

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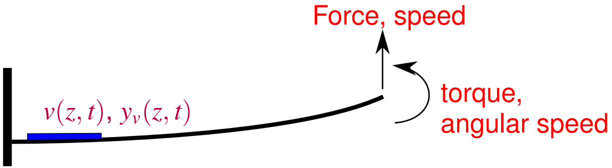
The piezoelectric beam equation can thus be rewritten as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -\partial_{z^2}^2 \\ \partial_{z^2}^2 & 0 \end{bmatrix}}_{\mathcal{J}} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} + \begin{bmatrix} \partial_{z^2}^2 \\ 0 \end{bmatrix} \Pi_{ab}(z) k_p v(z,t)\,,$$

where $\mathcal{J}$ is a formally skew-symmetric operator on $L^2(0,L)$.
The time-derivative of the Hamiltonian is computed as:

$$\begin{aligned} \dot{H} &= \int_{z=0}^{L} (e_1 \dot{x}_1 + e_2 \dot{x}_2)\, \mathrm{d}z\,, \\ &= \int_{z=0}^{L} \left(e_1 \left(-\partial_{z^2}^2 e_2 + \partial_{z^2}^2 \Pi_{ab}(z) k_p v(z,t)\right) + e_2\, \partial_{z^2}^2 e_1\right) \mathrm{d}z\,, \\ &= \left(-\underbrace{e_1}_{\text{speed}} \underbrace{\partial_z(e_2)}_{\text{force}} + \underbrace{\partial_z(e_1)}_{\text{angular vel.}} \underbrace{e_2}_{\text{torque}}\right)\Bigg|_{z=0}^{L} + \int_{z=a}^{b} \underbrace{k_p\, \partial_{z^2}^2 e_1}_{y_v(z,t)} v(z,t)\, \mathrm{d}z\,. \end{aligned}$$

Ex 4: Bending eqn as port-Hamiltonian system (3)



With distributed input port $v(z,t)$ and output port (conjugated to $v(z,t)$):

$$y_v(z,t) := k_p\, \partial_{z^2}^2 e_1\,, \quad a < z < b\,.$$

And boundary ports*:

$$\boldsymbol{y}_\partial := \begin{bmatrix} \partial_z \boldsymbol{e}_2(0) \\ -\boldsymbol{e}_2(0) \\ -\boldsymbol{e}_1(L) \\ \partial_z \boldsymbol{e}_1(L) \end{bmatrix} = \begin{bmatrix} \text{force} \\ \text{torque} \\ \text{vert. speed} \\ \text{angular speed} \end{bmatrix}, \; \boldsymbol{u}_\partial = \begin{bmatrix} \boldsymbol{e}_1(0) \\ \partial_z \boldsymbol{e}_1(0) \\ \partial_z \boldsymbol{e}_2(L) \\ \boldsymbol{e}_2(L) \end{bmatrix} = \begin{bmatrix} \text{vert. speed} \\ \text{angular speed} \\ \text{force} \\ \text{torque} \end{bmatrix}.$$

$$\dot{H} = \boldsymbol{y}_\partial^T \boldsymbol{u}_\partial + \int_{z=a}^{b} v(z,t) y_v(z,t)\, \mathrm{d}z\,.$$

* other possible choices of boundary ports: LE GORREC, ZWART, MASCHKE 2005

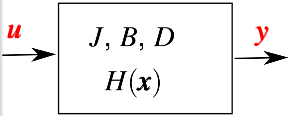
Overview

Summary on port-Hamiltonian Systems (pHs)

In finite-dimension:

$$\dot{\boldsymbol{x}} = J \nabla_{\boldsymbol{x}} H(\boldsymbol{x}) + B\boldsymbol{u}\,,$$
$$\boldsymbol{y} = B^T \nabla_{\boldsymbol{x}} H(\boldsymbol{x}) + D\boldsymbol{u}\,,$$

where J and D are skew-symmetric matrices.
We easily verify the **power-balance**: $\dot{H} = \boldsymbol{y}^T \boldsymbol{u}$.



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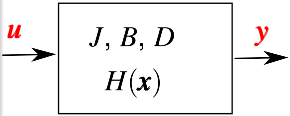

The infinite-dimensional case (in 1D):

$$\dot{\boldsymbol{x}}(z,t) = \mathcal{J}\,\delta_{\boldsymbol{x}} H(\boldsymbol{x}) + B\,u(z,t)\,,$$
$$y(z,t) = B^*\,\delta_{\boldsymbol{x}} H(\boldsymbol{x})\,,$$

with boundary control and observation:

$$\boldsymbol{u}_\partial = \mathcal{B}\boldsymbol{x}\,, \quad \boldsymbol{y}_\partial = \mathcal{C}\boldsymbol{x}$$

Power-balance: $\dot{H} = \boldsymbol{y}_\partial^T \boldsymbol{u}_\partial + \int u(z,t) y(z,t) dz$.

Summary on port-Hamiltonian Systems (pHs)

In finite-dimension:

$$\dot{\boldsymbol{x}} = J \nabla_{\boldsymbol{x}} H(\boldsymbol{x}) + B\boldsymbol{u}\,,$$
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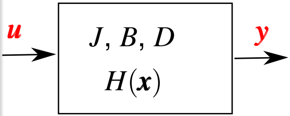

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Our interest for pHs:

- Interconnection and **modularity**;
- Obvious physical properties;
- Multiphysics;
- Energy-based control.

Summary

Port-Hamiltonian formulation provides a:

- modular,
- physically motivated (energy-based),
- multi-domain

framework for analyzing, simulating and controlling (complex) systems.

Also available (not presented today):

- Damping models, including thermodynamical consistency,
- Energy preserving simulation in the discrete-time domain,
- Geometric discretization of PDEs into ODEs in pHs form,
- Worked-out model and simulation a fluid-structure interaction (FSI) problem, like sloshing, see e.g. `http://github.com/flavioluiz/port-hamiltonian`
- Realistic PDEs in 2D or 3D domain, with boundary control and observation.

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